Midterm 2 - Solutions

You have until 1:50pm to complete the exam, be certain to use your time wisely. Answer all questions directly on the exam. You must show all of your work to receive full credit. Calculators may be used although you may leave answers as fractions. Unless a problem says otherwise, you can assume that firms can use fractions of units of inputs, produce fractions of units of output and charge non-integer prices (so a firm could use 28.6 units of input to produce 82.4 units and sell at a price of $5.325 per unit). Remember to put your name on the exam. Good luck!

Name: **ID Number:**

1. (20 points) For each of the scenarios below, write down a production function that matches the description of the firm's production technology.

   (a) A baker uses eggs (E), flour (F) and water (W) to produce loaves of bread (L). One loaf of bread requires exactly two eggs, four cups of flour and two cups of water. The baker cannot produce partial loaves of bread (it only has one size of loaf pan).

   The number of loaves the baker can produce will be limited by the ingredient that is in short supply. For example, if the baker has only two eggs there is no way to produce more than one loaf of bread regardless of how much flour and water is used. This type of production technology in which the output is restricted by whichever input is in short supply is captured by a min function of the following form:

   $$L = \min(\alpha E, \beta F, \gamma W)$$

   The task is to figure out what the coefficients α, β and γ should be. The coefficients need to match up with the amount of each input needed to produce a unit of output. Since one loaf of bread requires two eggs, the coefficient in front of E should give us $L = 1$ when $E = 2$. This tells us that α should equal $\frac{1}{2}$. Similar reasoning tells us that β should equal $\frac{1}{4}$ and γ should equal $\frac{1}{2}$. So our final production function is:

   $$L = \min\left(\frac{1}{2}E, \frac{1}{4}F, \frac{1}{2}W\right)$$

   (b) A tax preparation service uses accountants (A) and computers (C) to complete tax returns (R). Every extra computer used by the service increases the number of completed tax returns by the same amount as the previous computer. Every extra accountant increases the number of tax returns filed but by a smaller amount than the previous accountant did. Each accountant is more productive if there are more computers.

   We are given several relevant pieces of information. First, if every extra computer used by the service increases the number of completed tax returns by the same amount as the previous computer, we know that the marginal product of computers should not be a function of C. This is only possible if the production function is linear in C. Second, we know that each extra accountant leads to a smaller increase in output than the previous accountant, telling us that the marginal product of accountants is decreasing. This can be achieved with an exponent less than one on A in the production function. Finally, we are told that an accountant is more productive if there are more computers. This means that the marginal product of A should be a function of C. A and C must therefore appear in the same term in the production function. A production function consistent with all of these properties is:

$$R = A^{\frac{1}{2}}C$$

Note that $MP_A = \frac{1}{2}A^{-\frac{1}{2}}C$ is decreasing in A and increasing in C and that $MP_C = A^{\frac{1}{2}}$ does not vary with C.

(c) Everything is the same as in part (b) except that each accountant's productivity does not depend on the number of computers used by the service.

The difference now is that the marginal product of an accountant should not depend on C. This means that C should not appear in the same term as A. A should still have an exponent smaller than one to capture the diminishing marginal product of A and the exponent on C should still be equal to one to capture the constant marginal product of C. The following production function captures these properties:

$$R = A^{\frac{1}{2}} + C$$

(d) A textbook publisher uses printing presses (P) and computers (C) to produce textbooks (T). The marginal product of printing presses is diminishing and the marginal product of computers is also diminishing. However, the textbook publisher's technology exhibits increasing returns to scale. If the publisher uses eight printing presses and one computer, it can publish 400 textbooks.

The key features to match are the diminishing marginal products of both inputs (something we can capture with exponents less than one on each input) and the increasing returns to scale. If we put the inputs in separate terms, the fractional exponents needed for diminishing marginal products of each input would lead to decreasing returns to scale. However, if we put the inputs in the same term, we can get increasing returns to scale if the exponents sum to a number greater than one. So the general form for our production function will be:

$$T = \Theta P^{\alpha} C^{\beta}$$

Diminishing marginal products for both P and C require that α and β are both less than one. Increasing returns to scale requires that $\alpha + \beta$ is greater than one (if we scale up inputs by an amount λ, output will be scaled up by $\lambda^{\alpha+\beta}$). Let's set α and β to both be equal to $\frac{2}{3}$. The last thing we need to do is determine the value of Θ. We can do this using the information that eight printing presses and one computer produce 400 textbooks:

$$400 = \Theta \cdot 8^{\frac{2}{3}} \cdot 1^{\frac{2}{3}}$$

$$400 = 4\Theta$$

$$\Theta = 100$$

So our final production function is:

$$T = 100P^{\frac{2}{3}}C^{\frac{2}{3}}$$

2. (25 points) There are two types of firms that produce yoga mats, type A and type B. The cost functions for a firm of each type are given by:

$$C_A(y) = 10y + 10y^2 \tag{1}$$

$$C_B(y) = 5y + 10y^2 \tag{2}$$

where y is the number of yoga mats produced by the firm. There are currently 30 type A firms in the industry and 20 type B firms in the industry. The industry is perfectly competitive and there is free entry and exit of firms in the long run.

(a) Find expressions for average total costs ($AC(y)$), average variable costs ($AVC(y)$) and marginal costs ($MC(y)$) for each firm type.

First, the equations for a firm of type A:

$$AC_A(y) = \frac{C_A(y)}{y} = 10 + 10y$$

All of the terms in the cost function depend on y, so they are all part of variable cost making average variable cost identical to average total cost:

$$AVC_A(y) = \frac{VC_A(y)}{y} = 10 + 10y$$

Finally, marginal costs are given by the derivative of the cost function with respect to y:

$$MC_A(y) = \frac{dC_A(y)}{dy} = 10 + 20y$$

Doing the same for the type B firm gives us:

$$AC_B(y) = \frac{C_B(y)}{y} = 5 + 10y$$

$$AVC_B(y) = \frac{VC_B(y)}{y} = 5 + 10y$$

$$MC_B(y) = \frac{dC_B(y)}{dy} = 5 + 20y$$

(b) Find the supply equation ($S(p)$) for each firm type. Be certain to be explicit about the range of prices over which each firm type will produce.

Let's begin with a firm of type A. First we need to determine the shut down price for the firm. The firm will not produce if price is below average variable cost. This tells us that the firm will not produce at any prices below the point where marginal cost intersects average variable costs. The quantity corresponding to this point can be found by setting marginal cost equal to average variable cost:

$$MC_A(y) = AVC_A(y)$$

$$10 + 20y = 10 + 10y$$
$$20y = 10y$$

The only quantity that solves this equation is zero. Plugging this quantity back into the marginal cost curve will give us the shutdown price for firm type A:

$$p_{SD}^{A} = MC_A(0)$$
$$p_{SD}^{A} = 10 + 20 \cdot 0$$
$$p_{SD}^{A} = 10$$

So a type A firm will produce nothing when price is below \$10. When the price is above \$10, the firm will decide how much to produce by setting price equal to marginal cost:

$$p = MC_A(y)$$
$$p = 10 + 20y$$
$$y = \frac{1}{20}p - \frac{1}{2}$$

This gives us our final expression for the supply curve for a type a firm:

$$S_A(p) = \begin{cases} 0 & \text{if } p < 10 \\ \frac{1}{20}p - \frac{1}{2} & \text{if } p \geq 10 \end{cases}$$

Now we need to repeat the same steps for a firm of type B:

$$MC_B(y) = AVC_B(y)$$
$$5 + 20y = 5 + 10y$$
$$20y = 10y$$
$$y = 0$$
$$p_{SD}^{B} = MC_B(0)$$
$$p_{SD}^{B} = 5 + 20 \cdot 0$$
$$p_{SD}^{B} = 5$$
$$p = MC_B(y)$$
$$p = 5 + 20y$$
$$y = \frac{1}{20}p - \frac{1}{4}$$
$$S_B(p) = \begin{cases} 0 & \text{if } p < 5 \\ \frac{1}{20}p - \frac{1}{4} & \text{if } p \geq 5 \end{cases}$$

(c) Graph the short run industry supply curve and the long run industry supply curve. Label all slopes, intercepts and kinks with their numerical values.

The short run industry supply curve can be found by aggregating all of the individual firm supply curves. When we do this we need to be careful to take into account the shutdown prices of each firm type:

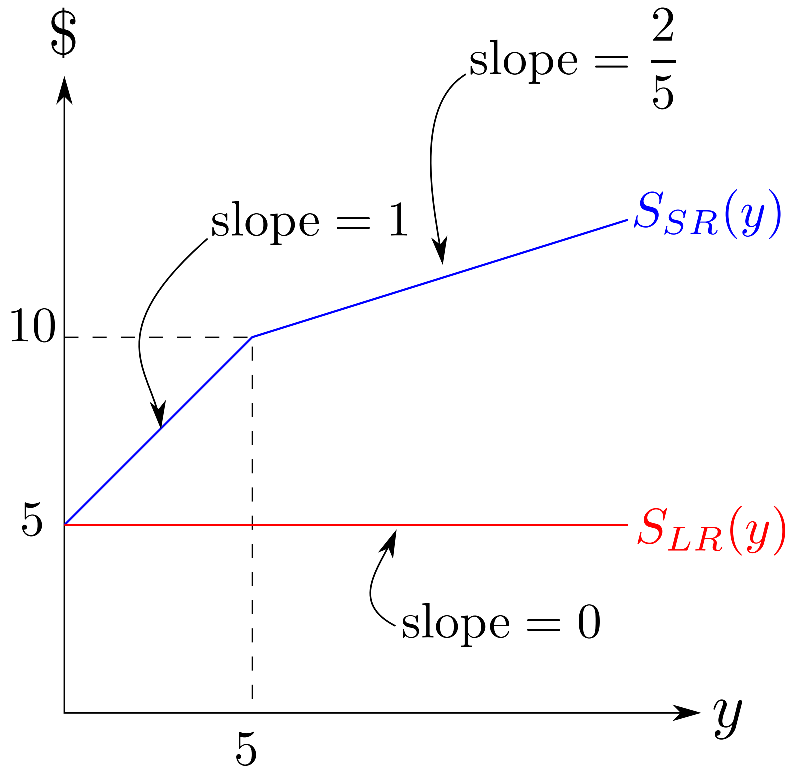
$$S_{SR}(p) = \begin{cases} 0 & \text{if } p < 5 \\ \sum_{i=1}^{20} S_B(p) & \text{if } 5 \leq p < 10 \\ \sum_{j=1}^{30} S_A(p) + \sum_{i=1}^{20} S_B(p) & \text{if } p \geq 10 \end{cases}$$

$$S_{SR}(p) = \begin{cases} 0 & \text{if } p < 5 \\ 20(\frac{1}{20}p - \frac{1}{4}) & \text{if } 5 \leq p < 10 \\ 30(\frac{1}{20}p - \frac{1}{2}) + 20(\frac{1}{20}p - \frac{1}{4}) & \text{if } p \geq 10 \end{cases}$$

$$S_{SR}(p) = \begin{cases} 0 & \text{if } p < 5 \\ p - 5 & \text{if } 5 \leq p < 10 \\ \frac{5}{2}p - 20 & \text{if } p \geq 10 \end{cases}$$

The long run supply curve will be at the lowest possible breakeven price. Notice that because there are no fixed costs for either firm type, the breakeven price will be the same as the shutdown price. So a type A firm will have a breakeven price of \$10 and a type B firm will have a breakeven price of \$5. In the long run, only B type firms will remain as entering firms will eventually drive price down to \$5, a price at which type A firms would lose money. The long run supply curve is simply a horizontal line at \$5.

The graph below shows the short run and long run industry supply curves. Note that the short run supply curve has a vertical intercept at \$5, the lower of the two shutdown prices, and a kink at \$10, the higher of the two shutdown prices.



(d) How would you expect your supply curves in part (c) to change if the prices of inputs for yoga mats fell? Be certain to fully explain your answer.

There are two possible ways lower input prices could change the decisions made by the firm.

If the prices of all inputs fell by the same amount such that the relative prices of inputs remained unchanged, the optimal combination of inputs would not change but the cost of that combination of inputs would be lower. So production would take place the same way it had before only with lower overall costs meaning there would also be lower marginal costs. This would lead to a new marginal cost curve for each firm down and to the right of the original marginal cost curve. Therefore the firms would be willing to supply more at any given price and would be willing to supply over a greater range of prices. This would lower the vertical intercept and kink of the short run supply curve and make both segments flatter. It would also shift the long run supply curve down as the breakeven price would now be lower.

The other possibility is that the reduction in input prices also led to change in the relative prices of inputs. This would lead firms to change the way they combine inputs, using more of the inputs that became relatively cheaper and less of the inputs that became relatively more expensive. Note that if the firm didn't change their combination of inputs its costs would still decline since input prices have fallen in absolute terms. If the firm reallocates its resources due to the relative price changes, this should lead to an even greater decrease in costs. So, once again, overall costs and marginal costs have fallen leading to all of the same changes in the supply curves discussed above.

3. (30 points) A farmer uses soil (S) and fertilizer (F) to grow zucchini (Z). The amount of zucchini the farmer can grow is given by the following production function:

$$Z = 10S + 4F^{\frac{1}{2}} \tag{3}$$

The price of a unit of soil is \$5. The price of a unit of fertilizer is \$2. The market price for a unit of zucchini is \$10.

(a) Find expressions for the marginal product of soil, the marginal product of fertilizer and the technical rate of substitution. Your expression for the technical rate of substitution should correspond to the slope of an isoquant on a graph with fertilizer on the horizontal axis and soil on the vertical axis.

First, the marginal products:

$$MP_S(S,F) = \frac{dZ(S,F)}{dS}$$

$$MP_S(S,F) = 10$$

$$MP_F(S,F) = \frac{dZ(S,F)}{dF}$$

$$MP_F(S,F) = \frac{2}{F^{\frac{1}{2}}}$$

The technical rate of substitution will be the ratio of the marginal products. However, we have to be careful to choose the right marginal product for the numerator and the right marginal product for the denominator. We want the technical rate of substitution to match up with the slope of an isoquant on a graph with F on the horizontal axis and S on the vertical axis. The slope of an isoquant on this graph is $\frac{\Delta S}{\Delta F}$. If the marginal product of soil is larger, it will take a larger ΔF to make up for any particular loss in soil ΔS. So our technical rate of substitution should be getting smaller as MP_S gets larger: MP_S should be in the denominator. Our technical rate of substitution is therefore:

$$TRS = -\frac{MP_F(F,S)}{MP_S(F,S)}$$

$$TRS = -\frac{\frac{2}{F^{\frac{1}{2}}}}{10}$$

$$TRS = -\frac{1}{5F^{\frac{1}{2}}}$$

(b) Suppose that in the short run, the farmer has 50 units of soil and is unable to buy any more for the season. The farmer can buy any amount of fertilizer he wants. If the farmer is maximizing profits, how much fertilizer will the farmer use and what profits will the farmer make?

In the short run with the level of soil fixed, the farmer will use any unit of fertilizer for which the revenue generated by the fertilizer exceeds the cost of the fertilizer. Therefore, the farmer will continue to use fertilizer up to the point where:

$$p_Z \cdot MP_F(S,F) = p_F$$
$$p_Z \frac{2}{F^{\frac{1}{2}}} = p_F$$
$$10 \cdot \frac{2}{F^{\frac{1}{2}}} = 2$$
$$F^{\frac{1}{2}} = 10$$
$$F = 100$$

So the farmer will use 100 units of fertilizer. Beyond this point, the cost of an additional unit of fertilizer will exceed the revenue the fertilizer generates. The profits of the farmer will be:

$$\pi = p_Z \cdot Z(S,F) - p_S \cdot S - p_Z \cdot Z$$
$$\pi = 10(10 \cdot 50 + 4 \cdot 100^{\frac{1}{2}}) - 5 \cdot 50 - 2 \cdot 100$$
$$\pi = 4950$$

(c) Suppose instead that the farmer has 100 units of fertilizer and cannot purchase any more this season. The farmer can purchase as much soil as he wants. If the farmer is maximizing profits, how much soil will the farmer want to use in the short run? Be certain to fully justify your answer.

The farmer will now want to use any unit of soil for which the revenue generated by the soil exceeds the cost of the soil. We could try to find this optimal level of soil using the same setup as above:

$$p_Z \cdot MP_S(S,F) = p_S$$
$$10 \cdot 10 = 5$$
$$100 = 5$$

Clearly this did not work. The problem is that the marginal product of soil is constant. So a unit of soil either always generates more revenues than it adds in costs or it always adds more in costs than it generates in revenues. In this case, the revenue generated by an extra unit of soil is \$100 (the left hand side of the equation above). The cost of an additional unit of soil is only \$5. So every extra unit of soil will always add to profits. The farmer will want to use as much soil as he can.

(d) Now consider the long run in which both soil and fertlizer are variable inputs. Derive expressions for the amount of fertilizer the farmer will use as a function of zucchini ($F(Z)$) and the amount of soil the farmer will use as a function of zucchini ($S(Z)$) assuming that the farmer minimizes costs. *Note: You can assume that the farmer is only considering producing large quantities of zucchini ($Z > 100$).*

To minimize costs, the farmer will want to adjust inputs until the ratio of the marginal products of the inputs is exactly equal to the ratio of their prices. If this were not the case, the farmer could save money by using more of the input that has a higher marginal product relative to its price and less of the input with a lower marginal product relative to its price. So the optimal combination of inputs should satisfy the following condition:

$$\frac{MP_S(S,F)}{MP_F(S,F)} = \frac{p_S}{p_F}$$

$$\frac{10}{\frac{2}{F^{\frac{1}{2}}}} = \frac{p_S}{p_F}$$

$$F^{\frac{1}{2}} = \frac{p_S}{5p_F}$$

$$F = \frac{p_S^2}{25p_F^2}$$

Plugging in our prices gives us the amount of F that will be used:

$$F = \frac{5^2}{25 \cdot 2^2}$$

$$F(Z) = \frac{1}{4}$$

We still don't have a value for the amount of soil the farmer will use. We can determine this using the amount of fertilizer found above and the production function:

$$Z = 10S + 4F^{\frac{1}{2}}$$

$$Z = 10S + 4(\frac{1}{4})^{\frac{1}{2}}$$

$$Z = 10S + 2$$

$$S(Z) = \frac{1}{10}Z - \frac{1}{5}$$

4. (25 points) The demand for gallons of gasoline (G) is given by the following inverse demand function:

$$p(G) = 10 - \frac{1}{10}G \tag{4}$$

The supply of gasoline is given by the following supply function:

$$S(G) = 10p \tag{5}$$

where p is the price per gallon of gasoline. Assume that the number of gas stations is fixed (there is no entry or exit of gas stations).

(a) Find the equilibrium price and quantity gasoline.

At the equilibrium price, supply should equal demand. Before we can set supply equal to demand we need to convert the inverse demand function into a demand function that gives quantity as a function of price:

$$p(G) = 10 - \frac{1}{10}G$$

$$p = 10 - \frac{1}{10}D(p)$$

$$\frac{1}{10}D(p) = 10 - p$$

$$D(p) = 100 - 10p$$

Now we can set demand equal to supply to solve for the equilibrium price:

$$D(p) = S(p)$$

$$100 - 10p = 10p$$

$$100 = 20p$$

$$p = 5$$

Plugging this back into either the demand or supply function will give us the equilibrium quantity:

$$D(5) = 100 - 10 \cdot 5$$

$$D(5) = 50$$

So the equilibrium price is \$5 and the equilibrium quantity is 50 gallons.

(b) The state decides to impose a 50% sales tax on gasoline for consumers. Consumers must pay this tax when filing their income tax returns (the tax is not collected by the gas station). What will the new equilibrium price for a gallon of gasoline at gas stations be (the price that is actually paid to the gas station) and what will the new equilibrium quantity of gasoline be?

Now, the equilibrium price will be the price for sellers at which the amount they are willing to supply is equal to the amount consumers demand at that price plus the tax:

$$D(p(1+\tau)) = S(p)$$

$$100 - 10p(1+\tau) = 10p$$

Plugging in the 50% tax rate gives us:

$$100 - 10p(1+.5) = 10p$$

$$25p = 100$$

$$p = 4$$

So the price received by gas stations will be \$4. The price paid by the consumers, including the tax, will be \$6 (\$4 plus the 50% tax of \$2). The new equilibrium quantity can be found by either plugging \$4 into the supply function or \$6 into the demand function. Either way you get a quantity of 40 gallons of gasoline.

(c) Calculate the tax revenue and the deadweight loss generated by this tax.

The tax revenue will be the \$2 in tax on each gallon (50% of the \$4 price) multiplied by the 40 gallons that are still sold, or \$80 in total tax revenue. The deadweight loss is the consumer and producer surplus that used to exist on the gallons of gas that are no longer sold. This is the area between the demand and supply curves between 40 gallons and 50 gallons:

$$DWL = \frac{1}{2}(50-40)(6-4)$$

$$DWL = 10$$

(d) How would your answers to (b) and (c) have changed if the gas stations were responsible for submitting the tax to the government? Be as specific as possible.

The only difference would be that the price that the gas station charges customers would be \$6 per gallon and then, after paying the tax, the gas station would be left with \$4 a gallon (assuming the 50% tax is still based on the net price received by the gas station, not the price before taxes are taken out). So from the customer and gas station's perspectives, nothing has changed. The consumer still ends up paying a dollar more, the gas station still ends up receiving a net price that is a dollar lower than before, the drop in quantity is the same and the amount of deadweight loss generated is therefore the same.