Midterm 1 - Solutions

You have until 4:50pm to complete the exam, be certain to use your time wisely. Answer all questions directly on the exam. You must show all of your work to receive full credit. Calculators may be used although you may leave answers as fractions. Unless a problem says otherwise, you can assume that individuals can consume fractions of units. Remember to put your name on the exam. Good luck!

Name: **ID Number:**

1. (25 points) Barry's preferences over gummy worms, G, and twizzlers, T, are given by the following utility function:

$$U(G,T) = 3G^{\frac{2}{3}} + 6T^{\frac{2}{3}} \tag{1}$$

(a) Determine whether Barry's preferences are convex or not. Be certain to fully justify your answer.

The simplest way to check whether his preferences are convex or not is to see if his marginal rate of substitution is decreasing. A decreasing marginal rate of substitution would imply convex preferences (as you reach extreme bundles, you are willing to trade large quantities of the good you have a lot of to get a small increase in the good you have very little of, you want to move from extreme to average bundles). To get Barry's marginal rate of substitution, we first need to get the marginal utility of each good:

$$MU_G = \frac{dU(G,T)}{dG} = \frac{2}{3} \cdot 3G^{-\frac{1}{3}} + 0 = 2G^{-\frac{1}{3}}$$

$$MU_T = \frac{dU(G,T)}{dT} = 0 + \frac{2}{3} \cdot 6T^{-\frac{1}{3}} = 4T^{-\frac{1}{3}}$$

The marginal rate of substitution can now be found using the ratio of these marginal utilities:

$$MRS = \frac{\Delta G}{\Delta T} = -\frac{MU_T}{MU_G}$$

$$MRS = -\frac{4T^{-\frac{1}{3}}}{2G^{-\frac{1}{3}}} = -2\frac{G^{\frac{1}{3}}}{T^{\frac{1}{3}}}$$

Notice that as you move to the right along an indifference curve, increasing T and decreasing G, this marginal rate of substitution is decreasing in magnitude. This implies convex indifference curves and convex preferences.

(b) Derive an expression for the optimal number of gummy worms as a function of income, the price of a gummy worm and the price of a twizzler, $G(I, p_G, p_T)$.

We can begin by using our tangency condition stating that at the optimal bundle the slope of the budget line will be equal to the slope of the indifference curve passing through that bundle. If this weren't the case, it would be possible to move along the budget line and reach a higher indifference curve. Setting the slope of the budget line equal to the slope of the indifference curve gives us:

$$-\frac{p_T}{p_G} = MRS$$

$$-\frac{p_T}{p_G} = -2\frac{G^{\frac{1}{3}}}{T^{\frac{1}{3}}}$$

$$T^{\frac{1}{3}} = 2\frac{p_G}{p_T}G^{\frac{1}{3}}$$

$$T = 8\frac{p_G^3}{p_T^3}G$$

Now we can plug this expression into the budget equation to find G in terms of prices and income:

$$p_T T + p_G G = I$$

$$p_T \cdot 8\frac{p_G^3}{p_T^3}G + p_G G = I$$

$$G\left(8\frac{p_G^3}{p_T^2} + p_G\right) = I$$

$$G = \frac{I}{8\frac{p_G^3}{p_T^2} + p_G}$$

(c) Determine whether twizzlers are a normal or inferior good and whether they are a substitute or complement for gummy worms. Be certain to fully justify your answer.

The simplest way to go about this is to first derive the demand equation for twizzlers. To do this, we can plug our demand equation for gummy worms back into our final equation for the tangency condition above:

$$T = 8\frac{p_G^3}{p_T^3}G$$

$$T = 8\frac{p_G^3}{p_T^3}\frac{I}{8\frac{p_G^3}{p_T^2} + p_G}$$

$$T = \frac{I}{p_T + \frac{p_T^3}{8p_G^2}}$$

From this demand equation, we can see that as I increases, T increases. Therefore twizzlers are a normal good. We can also see that as p_G increases, the denominator in the above equation gets smaller, making T larger. So twizzlers are a substitute for gummy worms.

(d) Suppose that gummy worms cost one dollar each and twizzlers cost two dollars each. Graph the Engel curve for gummy worms being certain to label all intercepts and slopes with their numerical values.

Plugging in the given prices into the demand equation for gummy worms gives us:

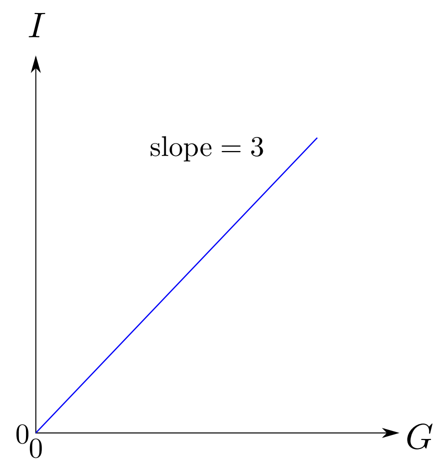
$$G = \frac{I}{8\frac{1^3}{2^2} + 1}$$

$$G = \frac{I}{3}$$

The Engel curve has income on the vertical axis and gummy worms on the horizontal axis, so it is useful to rearrange this equation to get I on the left-hand side:

$$I = 3G$$

This is now the equation for our Engel curve. It is a straight line with a slope of three passing through the origin.



2. (25 points) Aaron enjoys watching both half-hour sitcoms and one-hour dramas. These are the only two types of shows that Aaron watches. During a typical week Aaron has ten hours to watch television. On weeks when he has more time, he watches both more sitcoms and more dramas.

(a) Write down a budget equation giving the affordable bundles of sitcoms, S, and dramas, D, for a typical week of TV viewing. S and D should be the only variables in your equation, everything else should be numerical constants. Show this budget line on a graph with sitcoms on the horizontal axis and dramas on the vertical axis. Be certain to label the intercepts and slope with their numerical values.

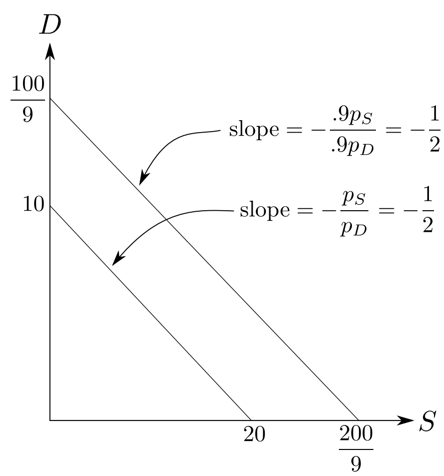
The constraint here is not income in dollars but rather how much time Aaron has. Each sitcom has a price of half an hour while each drama has a price of one hour. If we think of Aaron's total time endowment as T, his budget equation would be:

$$p_S S + p_D D = T$$

where the prices are denominated in hours not dollars. Plugging in the appropriate numbers gives us:

$$\frac{1}{2} \cdot S + 1 \cdot D = 10$$

The graph of this budget line is shown below:



(b) Suppose that Aaron purchases a DVR that allows him to skip commercials. Commercials make up ten percent of the total running time of both sitcoms and dramas. On the same graph as part (a), show how this changes Aaron's budget line. Once again, be certain to label all intercepts and slopes of the new budget line with their numerical values.

The DVR effectively reduces the price of each show. Instead of each sitcom taking half an hour, it now takes only 90 percent of that time. Instead of a drama taking a full hour, it now takes only 90 percent of an hour. This changes our budget equation to:

$$\frac{9}{10} p_S S + \frac{9}{10} p_D D = 10$$

$$\frac{9}{10} \cdot \frac{1}{2} \cdot S + \frac{9}{10} \cdot 1 \cdot D = 10$$
$$\frac{9}{20} S + \frac{9}{10} D = 10$$

This new budget line is shown on the graph above. Note that the new budget line has shifted out but that the slope of the budget line remains the same as the relative prices have not changed (both prices decreased by the same percentage).

(c) Will the purchase of the DVR increase or decrease the number of sitcoms Aaron watches and will it increase or decrease the number of dramas he watches? Be certain to fully explain your answer. If the answer is ambiguous, explain what additional information you would need to determine the direction of change.

From the previous part, we can see that by reducing both prices by the same percentage, the net effect of the DVR was to increase Aaron's effective income (his time). The relative prices of the shows has not changed. Given this effective increase in income, whether each type of show is normal or inferior will determine whether the number watched increases or decreases. In the initial statement of the problem, we are told that Aaron watches more of both types of shows if he has more time. This tells us that both shows are normal goods. When the DVR increases Aaron's effective income, he will consume more of both.

(d) Suppose that the impact of an additional sitcom on Aaron's utility is bigger if he is watching a lot of dramas (he appreciates the break from all of the seriousness of the dramas). The impact of an additional sitcom on his utility is smaller if he is watching very few dramas. The impact of an additional drama on his utility does not depend on the number of dramas he is currently watching. Write down a utility function that is consistent with Aaron's preferences. Your function should contain only the number of sitcoms, S, the number of dramas, D, and numerical constants. It shouldn't include any other variables or parameters.

We are being told two different things about the marginal utilities: (i) MU_S increases as D increases and (ii) MU_D does not depend on D. For our utility function to produce property (i), S and D must appear in the same term (otherwise when taking a derivative with respect to S to get MU_S, D will disappear). For our utility function to produce property (ii), D must disappear when taking the derivative of the utility function with respect to D. The only way for this to happen is if utility is a linear function of D. A generic utility function satisfying these conditions is the following:

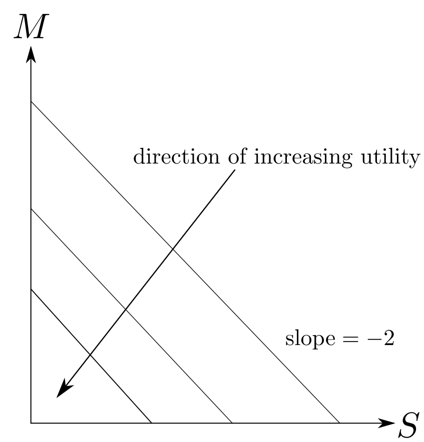
$$U(S, D) = S^a D$$

The value you choose for the exponent on S must be positive since sitcoms are a good but can be less than, equal to or greater than one because we are not told anything about diminishing, constant or increasing marginal utility of S.

3. (20 points) For each scenario below, draw three indifference curves consistent with the description of the preferences over the two goods in question. Where possible, include numerical values for slopes. Be certain to clearly label which axis is which and indicate the direction in which utility is increasing.

(a) Candice hates mice, M, and she hates spiders, S. The effect of each spider on Candice's utility is twice as large as the effect of each mouse regardless of how many mice and spiders she currently has.

In this case, both of the goods are actually bads. With one good and one bad, indifference curves are upward sloping. However, with two bads indifference curves are downward sloping just like standard indifference curves only the direction of increasing utility is toward the origin. In this case, the marginal utility of a spider is always double the marginal utility of a mouse, so the marginal rate of substitution will always be equal to -2 on a graph with spiders on the horizontal axis. A constant marginal utility implies a constant slope for the indifference curves.

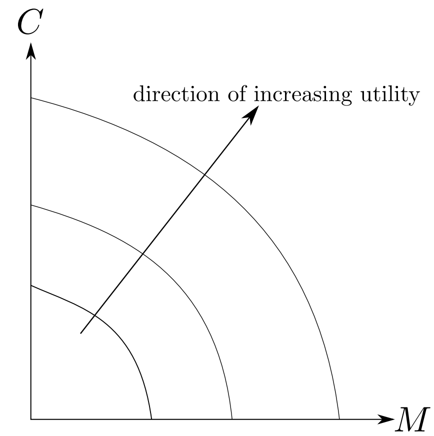


(b) Darla likes both milk, M, and cookies, C. The marginal utility Darla gets from each glass of milk is 5 utils (units of utility). The marginal utility of cookies is increasing as the number of cookies increases.

Milk and cookies are both goods, so the indifference curves will be downward sloping. The marginal rate of substitution for a graph with M on the horizontal axis will be:

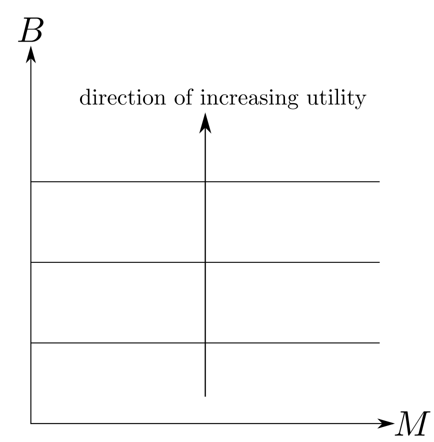
$$MRS = -\frac{MU_M}{MU_C} = -\frac{5}{MU_C}$$

As we move down and right along an indifference curve, increasing M and decreasing C, the numerator of the MRS stays the same but the denominator is increasing (MU_C decreases as C decreases). So overall the MRS is getting larger in magnitude telling us that the indifference curves get steeper going from left to right.



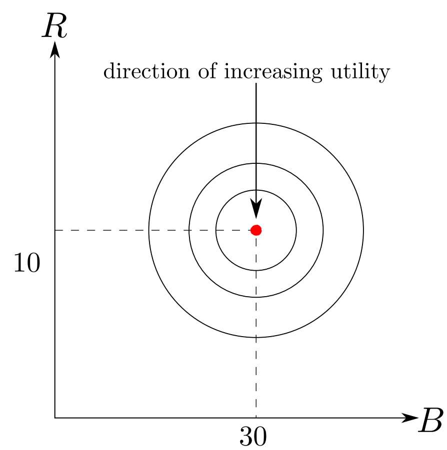
(c) Elizabeth likes reading books, B. Each additional book always increases Elizabeth's utility but at a diminishing rate. Elizabeth does not read magazines, M. Having more magazines has no impact on her utility.

The fact that magazines have no impact on Elizabeth's utility tells us that her indifference curves on a graph with magazines on the horizontal axis will be horizontal lines. To see why this is the case, think about starting at any paricular bundle of books and magazines. If you add a magazine to this bundle, utility doesn't change so that new bundle should be on the same indifference curve. Likewise, if you take away a magazine, utility doesn't change and you are still on the same indifference curve. The only way to move to a different indifference curve is to add or subtract books.



(d) Frederick likes to run and bike. Every mile of running, R, up to ten miles increases Frederick's utility. After that, Frederick gets tired and additional miles actually reduce his utility. The same thing is true of biking miles, B, except that additional biking increases Frederick's utility up to thirty miles and then decreases it after that.

This is similar to the sushi buffet in Problem Set 1 only now Frederick has a satiation point for both goods. To picture the indifference curves on a graph with biking miles on the horizontal axis, think about splitting the graph up into four quadrants: (i) points down and left of thirty biking miles and ten running miles, (ii) points up and left of that point, (iii) points up and right of that point, and (iv) points down and right of that point. In quadrant (i), Frederick wants to run more and bike more, making both good and giving us a downward sloping indifference curve. In quadrant (ii) Frederick wants to bike more but run less, making biking a good but running a bad, giving us upward sloping indifference curves. In quadrant (iii), both are bads giving us downward sloping indifference curves and finally in quadrant (iv) running is good but biking is bad giving us upward sloping indifference curves. So the indifference curves will actually be closed shapes centered around the point (30 biking miles, 10 running miles) that have negatively sloped sides in quadrants (i) and (iii) and positively sloped sides in quadrants (ii) and (iv). A simple shape that accomplishes this would be a circle.



4. (30 points) Gary's utility from football tickets, F, and money spent on other things, M, is given by the following utility function:

$$U(F, M) = 20F^{\frac{1}{2}} + M \tag{2}$$

The price of a football ticket is \$2 and Gary has \$200 to spend.

(a) Determine the number of football tickets Gary will buy.

We can begin with our tangency condition stating that the slope of the budget line should be equal to the slope of the indifference curve at the optimal bundle:

$$-\frac{p_F}{p_M} = MRS$$

$$-\frac{p_F}{p_M} = -\frac{MU_F}{MU_M}$$

Using the appropriate derivatives of the utility function to get the marginal utilities, this simplifies to:

$$\frac{p_F}{p_M} = \frac{10F^{-\frac{1}{2}}}{1}$$

Note that the price of a unit of money spent on other goods is simply \$1 (\$1 worth of other stuff costs \$1). So our tangency condition reduces to:

$$p_F = 10F^{-\frac{1}{2}}$$

$$F = \frac{100}{p_F^2}$$

Now we can plug in the price of a football ticket, \$2, to get the optimal number of tickets:

$$F = \frac{100}{2^2} = 25$$

So Gary will buy 25 football tickets and then spend his remaining \$150 on other things.

(b) Graph Gary's price offer curve, labeling the points that correspond to prices of football tickets equal to \$0.25, \$0.50, \$1, and \$2.

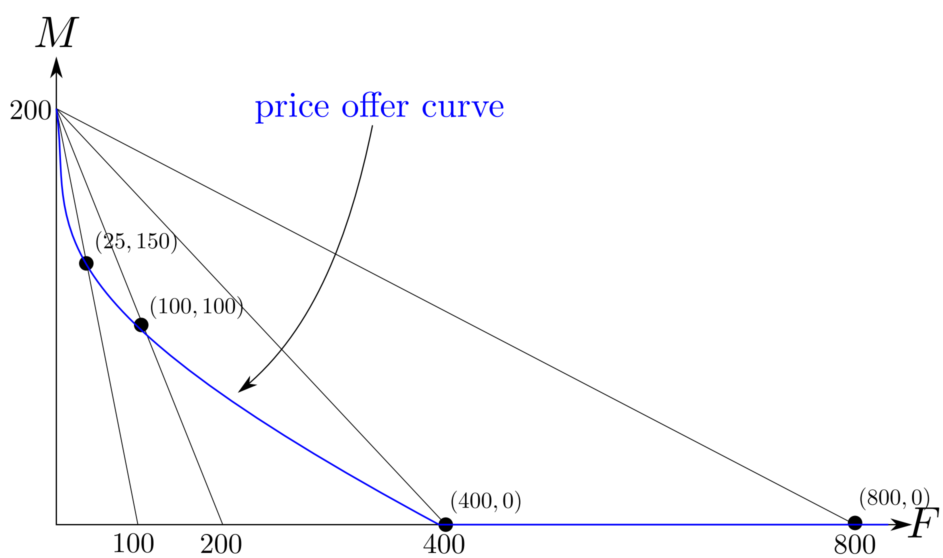
The price offer curve simply shows the optimal combinations of football tickets and money spent on other things as the budget line rotates based on the different prices of football tickets. We already found an equation to get the optimal number of football tickets above. The money spent on other things at the optimal bundle is simply whatever is leftover from our income after buying tickets:

$$M = I - p_F \cdot F$$

$$M = I - p_F \frac{100}{p_F^2}$$

$$M = 200 - \frac{100}{p_F}$$

Now we can see that we have a bit of a problem. When the price of a football ticket drops below \$0.50, Gary will try to spend more than \$200 on football tickets and a negative amount on other things. This is not possible. So, for prices below \$0.50, Gary will be stuck at a corner solution, spending all of his money on football tickets. This means that the price offer curve will look very different for prices above \$0.50 and prices below \$0.50. At prices above \$0.50, decreasing the price of a football ticket will increase the number of tickets purchased and decrease the money spent on other things (from above we see that M is a decreasing function of p_F and F is an increasing function of p_F). This will give us a downward sloping price offer curve. At prices below \$0.50, Gary will spend all of his money on football tickets so the price offer curve will be tracing out the horizontal axis.



(c) Graph Gary's demand curve for football tickets, labeling any slopes and kinks with there numerical values where possible.

Using the information from above, we know that the demand curve is going to have two different segments, one for prices above \$0.50 and one for prices below \$0.50. On the first segment, demand is given by $F = \frac{100}{p_F^2}$. From this demand function we can see that as p_F goes to infinity, F will approach zero. So as price increases the demand curve will approach the vertical axis. So the first portion of the demand curve will be a downward sloping convex curve approaching infinity as F goes to zero and ending at $F = 400$ when price falls to \$0.50. The second segment of the demand curve corresponds to the range of prices in which Gary is spending all of his income on tickets. Therefore his demad for tickets is given by $F = \frac{I}{p_F}$. This is a downward sloping, convex curve that approaches the horizontal axis as price approaches zero.

There is one last thing to consider before graphing the demand curve. We know there is a kink at $F = 400$ but we don't know yet whether the demand curve is steeper to the left or to the right of the kink. We can figure this out by looking at the derivative of the demand curve at $F = 400$ using the two different demand functions. (*Note: I did not expect you to do this part on the exam.*). First, we need to rewrite the demand equations as price as a function of quantity since quantity is on the horizontal axis:

$$p_F^A = \frac{10}{F^{\frac{1}{2}}}$$

$$p_F^B = \frac{200}{F}$$

Now we can find take the derivative with respect to F to find the slopes of each segment:

$$\frac{dp_F^A}{dF} = -\frac{5}{F^{\frac{3}{2}}}$$

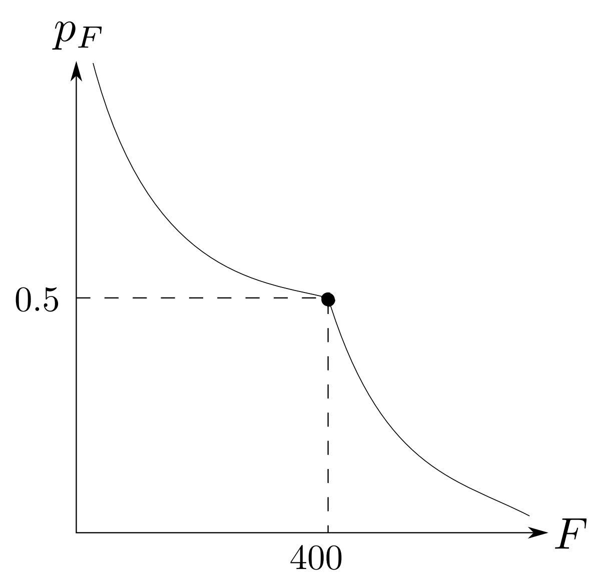
$$\frac{dp_F^B}{dF} = -\frac{200}{F^2}$$

Plugging in $F = 400$ gives us the slope of each segment at the kink:

$$\frac{dp_F^A}{dF} = -\frac{5}{400^{\frac{3}{2}}} = -0.000625$$

$$\frac{dp_F^B}{dF} = -\frac{200}{400^2} = -0.00125$$

So the segment of the demand curve to the left of the kink is flatter than the segment to right at $F = 400$. We now have a complete picture of the demand curve.



(d) Show that the utility function of $V(F, M) = 400F + M^2 + 40F^{\frac{1}{2}}M$ also represents Gary's preferences.

First note that this new utility function is increasing in both F and M, so both F and M are goods just as they were in the original utility function. Given that, if the two utility functions lead to the same marginal rate of substitution both would generate the same exact indifference curves and represent the same preferences. The marginal rate of substitution using this new utility function is:

$$MRS = -\frac{MU_F}{MU_M}$$

$$MRS = -\frac{\frac{dV(F,M)}{dF}}{\frac{dV(F,M)}{dM}}$$

$$MRS = -\frac{400 + 20F^{-\frac{1}{2}}M}{2M + 40F^{\frac{1}{2}}}$$

We know that we are trying to get this to look like $10F^{-\frac{1}{2}}$ so let's pull out that term from the numerator:

$$MRS = -\frac{(10F^{-\frac{1}{2}})(40F^{\frac{1}{2}} + 2M)}{2M + 40F^{\frac{1}{2}}}$$

This reduces to exactly what we were looking for:

$$MRS = -\frac{10}{F^{\frac{1}{2}}}$$

So the utility function $V(F, M)$ would also represent Gary's preferences.